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Quasi Topologies and Rational Approximation

THOMAS BAGBY

Indiana University, Bloomington, Indiana 47401

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We obtain a characterization for L_p approximation by analytic functions on compact plane sets which is analogous to Vitushkin's characterization for uniform approximation. For $p = 2$ this was done by Havin by use of Cartan's fine topology; we study the general case by use of quasi topologies.

1. INTRODUCTION

In this paper we discuss quasi topologies and their application to problems of mean approximation by complex analytic functions in the plane $\mathbf{C}$. If $X \subset \mathbf{C}$ is a compact set of positive Lebesgue measure, we denote by $L_p(X)$ the L_p space obtained from Lebesgue measure restricted to X. We wish to study the subspace $L_p{}^a(X)$ consisting of all functions in $L_p(X)$ which are analytic at every point of the interior int X.

THEOREM 1. *If $X \subset \mathbf{C}$ is a compact set of positive Lebesgue measure, $1 < p < \infty$ and $1/p + 1/q = 1$, then the following are equivalent*:

(a) *The rational functions with poles off X are dense in $L_p{}^a(X)$.*

(b) $\gamma_q(G - X) = \gamma_q(G - \operatorname{int} X)$ *for every bounded open set* $G \subset \mathbf{C}$.

Here the q-capacity γ_q is defined as follows. If $K \subset \mathbf{C}$ is compact we let

$$\gamma_q(K) = \inf_u \| u \|_q ,$$

where

$$\| u \|_q = \left\{ \iint [| u |^2 + | \operatorname{grad} u |^2]^{q/2} \, dx \, dy \right\}^{1/q}$$

and the infimum is taken over all real-valued functions $u \in C_0^\infty(\mathbf{C})$ such that $u = 1$ on K. If $E \subset \mathbf{C}$ is arbitrary we define the capacity

$$\gamma_q(E) = \sup\{\gamma_q(K) : K \subset E, K \text{ compact}\}.$$

If $1 \leqslant p < 2$, it is well known that (a) holds for every compact set X; for $p \geqslant 2$ this is no longer true. If $p = 2$, the above theorem is due to Havin [9], who worked with the fine topology of potential theory. In the present paper we use quasi topological concepts [8], which are discussed in Sections 2 and 3. The proof of Theorem 1 is given in Section 4. The proof is much simpler when X has no interior points; this proof was given earlier by the writer [1].

Mean approximation by rational functions has been studied by a number of writers, including Brennan [2], Hedberg [10], and Sinanjan [14].

2. Sobolev Spaces and Capacity

If $1 < q < \infty$, we denote by W_q^1 the Banach space of all functions $u \in L_q(\mathbf{R}^n)$ whose first partial derivatives (in the sense of distribution theory) are also in $L_q(\mathbf{R}^n)$, the norm being defined by

$$\| u \| = \| u \|_q = \left\{ \int [| u |^2 + | \nabla u |^2]^{q/2} \, d\lambda \right\}^{1/q}.$$

Here ∇u denotes the gradient of u, and λ denotes Lebesgue measure in $\mathbf{R}^n$. The basic facts about these spaces may be found in [13, Chapter 3].

Now let Ω be a fixed open set in $\mathbf{R}^n$. We denote by $W_{q0}^1(\Omega)$ the closure of $C_0^\infty(\Omega)$ in W_q^1. We define the *capacity* γ_Ω of a compact set $E \subset \Omega$ to be

$$\gamma_\Omega(E) = \inf_u \| u \|, \tag{1}$$

where the infimum is taken over all real-valued functions $u \in C_0^\infty(\Omega)$ such that $u = 1$ on E. We note that the infimum (1) may be taken over all real-valued functions $u \in C_0^\infty(\Omega)$ such that $u \geqslant 1$ on E; this can be proved by truncation and use of mollifiers [13]. For an arbitrary set $E \subset \Omega$ we define the capacity

$$\gamma_\Omega(E) = \sup\{\gamma_\Omega(K) : K \subset E \text{ compact}\}$$

and the exterior capacity

$$\gamma_\Omega{}^*(E) = \inf\{\gamma_\Omega(G) : G \supset E \text{ open}\}.$$

A set $E \subset \Omega$ is *γ_Ω-capacitable* if $\gamma_\Omega(E) = \gamma_\Omega{}^*(E)$.

From the definitions we see that every open set in Ω is γ_Ω-capacitable. Moreover, for any decreasing sequence of compact sets $K_j \subset \Omega$ we have $\gamma_\Omega(K_j) \to \gamma_\Omega(\bigcap K_j)$. It follows that every compact set in Ω is γ_Ω-capacitable.

From the subadditivity of the norm we get

$$\gamma_\Omega(K_1 \cup K_2) \leqslant \gamma_\Omega(K_1) + \gamma_\Omega(K_2)$$

for compact sets $K_1, K_2 \subset \Omega$. From this follows the countable subadditivity

$$\gamma_\Omega{}^* \left(\bigcup_{j=1}^{\infty} E_j\right) \leqslant \sum_{j=1}^{\infty} \gamma_\Omega{}^*(E_j),$$

for any sequence of sets $E_j \subset \Omega$.

If K is a compact subset of two open sets Ω_1 and Ω_2, it follows from Leibniz' rule that there exists a positive constant $C = C(K, \Omega_1, \Omega_2)$ such that

$$C^{-1}\gamma_{\Omega_1}(E) \leqslant \gamma_{\Omega_2}(E) \leqslant C\gamma_{\Omega_1}(E), \qquad \text{if} \quad E \subset K,$$

$$C^{-1}\gamma^*_{\Omega_1}(E) \leqslant \gamma^*_{\Omega_2}(E) \leqslant C\gamma^*_{\Omega_1}(E), \qquad \text{if} \quad E \subset K.$$

As a consequence of this and the countable subadditivity, the statement that a set $E \subset \Omega$ satisfies $\gamma_\Omega{}^*(E) = 0$ is independent of the containing open set Ω. A property is said to hold *quasi everywhere* (q.e.) if the set where it fails has zero exterior capacity.

The following lemma of Egorov type follows from an argument of Deny and Lions (see [7, Chapter II, Theorem 3.1] or [17, Theorem 4.3]).

LEMMA 1. *Suppose that the functions $u_j \in C_0{}^\infty(\mathbf{R}^n)$ form a convergent sequence in $W_q{}^1$. For every $\epsilon > 0$ there exists an open set W with $\gamma_{\mathbf{R}^n}(W) < \epsilon$ and a subsequence of $\{u_j\}$ converging uniformly off W.*

This lemma motivates the following definitions. A set $E \subset \mathbf{R}^n$ is said to be *quasi open* (resp. *quasi closed*) if for every $\epsilon > 0$ there exists an open set $W \subset \mathbf{R}^n$ with $\gamma_{\mathbf{R}^n}(W) < \epsilon$ such that $E - W$ is open (resp. closed) in $\mathbf{R}^n - W$. A function $f: \mathbf{R}^n \to \mathbf{C}$ is *quasi continuous* if for every $\epsilon > 0$ there exists an open set $W \subset \mathbf{R}^n$ with $\gamma_{\mathbf{R}^n}(W) < \epsilon$ such that the restriction of f to $\mathbf{R}^n - W$ is continuous. By use of mollifiers, it follows from Lemma 1 that every function in W_q^1 coincides a.e. with a quasi continuous function. If f is any quasi continuous function and $G \subset \mathbf{C}$ is open, then the inverse image $f^{-1}(G)$ is quasi open. Every quasi open set can be written as the union of a Borel set and a set of zero exterior capacity.

The basic facts concerning the capacity γ_Ω and the quasi continuous functions $u \in W_q^1$ are summarized in the following Theorem 2. Using slightly different definitions, Ziemer [17] proved such a theorem by means of the theory of extremal length. For completeness we include here a direct proof.

THEOREM 2. (i) *If E is any subset of an open set $\Omega \subset \mathbf{R}^n$, then*

$$\gamma_\Omega(E) \leqslant \inf_{u \in \mathscr{Q}(E,\Omega)} \| u \| \leqslant \gamma_\Omega^*(E),$$

where $\mathscr{Q}(E, \Omega)$ is the class of all real-valued quasi continuous functions $u \in W_{q0}^1(\Omega)$ such that $u \geqslant 1$ q.e. *on E.*

(ii) *Every Suslin subset of an open set $\Omega \subset \mathbf{R}^n$ is γ_Ω-capacitable. In particular, for Suslin sets E the inequalities in part* (i) *are both equalities.*

(iii) *If u and v are quasi continuous functions in W_q^1 which represent the same distribution, then $u = v$* q.e. *in $\mathbf{R}^n$. In particular, the mollifiers of a quasi continuous function $u \in W_q^1$ converge pointwise to u* q.e. *in $\mathbf{R}^n$.*

Remark. If $E = W$ is an open subset of an open set Ω and $\gamma_\Omega(W) < \infty$, we shall show that the infimum in part (i) is assumed by a quasi continuous function $w \in W_{q0}^1(\Omega)$ such that $w \equiv 1$ on W. We call w an *equilibrium potential* for W.

Proof. The first statement of part (ii) follows from the theorem of Choquet [6; 3], using the easily proved fact that $\Gamma_q = [\gamma_q]^q$ satisfies the inequality

$$\Gamma_q(K_1 \cap K_2) + \Gamma_q(K_1 \cup K_2) \leqslant \Gamma_q(K_1) + \Gamma_q(K_2)$$

for any pair of compact sets $K_1, K_2 \subset \Omega$.

We turn now to the remark. Let W be an open subset of Ω. Let $K_j = \{x \in W: d(x, \partial W) \leqslant j^{-1} \text{ and } d(x, 0) \leqslant j\}$. Since $\gamma_\Omega(K_j) \to \gamma_\Omega(W)$, we can find functions $u_j \in C_0^\infty(\Omega)$ such that $u_j = 1$ near K_j and $\| u_j \| \to \gamma_\Omega(W) < \infty$. From the elementary theory of Sobolev spaces

[13] it follows that some subsequence of $\{u_j\}$, which we still call $\{u_j\}$, converges weakly to $u \in W^1_{q0}(\Omega)$. Then $u = 1$ a.e. on W, and by lower semicontinuity of the norm we conclude that $\| u \| \leqslant \gamma_\Omega(W)$. Now let $w \in W^1_{q0}(\Omega)$ be the pointwise limit of a sequence of mollifiers of u; then $w = 1$ everywhere on W and $\| w \| \leqslant \gamma_\Omega(W)$. This last inequality is in fact an equality, and this will follow from statement (i).

We now prove the following special case of (i): if $E \subset \mathbf{R}^n$ is compact, and a real-valued quasi continuous function $u \in W_q{}^1$ satisfies $u \geqslant 1$ on E, then $\gamma(E) \leqslant \| u \|$. It is no loss of generality to assume $u \geqslant 0$ everywhere. If $\epsilon > 0$ is arbitrary, we let W be an open set such that u is continuous on $\mathbf{R}^n - W$ and $\gamma(W) < \epsilon$. It is clear that we can find a bounded open set G containing E such that $u \geqslant 1 - \epsilon$ on $G - W$. If w is an equilibrium potential for W, $v = (u + w)/(1 - \epsilon)$, and $\phi \in C_0{}^\infty$ is chosen so that $\phi \equiv 1$ on $\bar{G}$ and $\| \phi v - v \| < \epsilon$, then by a mollifier argument we have

$$\gamma(E) \leqslant \| \phi v \| \leqslant \| v \| + \epsilon \leqslant (\| u \| + \epsilon)/(1 - \epsilon) + \epsilon.$$

Since ϵ was arbitrary, we obtain $\gamma(E) \leqslant \| u \|$, as required.

To prove assertion (iii) we let $u \in W_q{}^1$ be any non-negative quasi continuous function which satisfies $\| u \| = 0$, and suppose the (capacitable) set $\{u > 0\}$ has positive capacity. Then for some $\epsilon > 0$ the (capacitable) set $\{u > \epsilon\}$ has positive capacity, so that we have $u > \epsilon$ on some compact set of positive capacity. This contradicts the result in the preceding paragraph, so (iii) is proved.

To prove the left-hand inequality of part (i), we may assume E is compact. We let $u \in \mathcal{Q}(E, \Omega)$ and select a sequence of functions $u_j \in C_0{}^\infty(\Omega)$ such that $u_j \to u$ in $W_q{}^1$. If $\epsilon > 0$ is arbitrary, then by (iii) we eventually have $\| u_j \| < \| u \| + \epsilon$ and $u_j > 1 - \epsilon$ on $E - W$, where W is an open set with $\gamma_\Omega(W) < \epsilon$. We obtain the estimate

$$\gamma_\Omega(E) \leqslant \gamma_\Omega(E - W) + \gamma_\Omega(W) \leqslant (\| u \| + \epsilon)/(1 - \epsilon) + \epsilon,$$

and since ϵ was arbitrary we have $\gamma_\Omega(E) \leqslant \| u \|$, which proves the left-hand inequality of (i). The right-hand inequality follows from the definition of exterior capacity and the existence of equilibrium potentials for open sets.

3. Null Sets of Quasi Continuous Functions

We turn now to a detailed study of the null sets of quasi continuous functions. From now on we will be interested in the capacity $\gamma = \gamma_{\mathbf{R}^n}$

relative to $\mathbf{R}^n$; the more general capacities γ_Ω will be used only as a technical convenience in some of the proofs.

THEOREM 3. *If K and L are disjoint sets of positive capacity, such that K is compact and L is quasi closed, then there exists a quasi continuous function in W_q^1 which is equal to zero* q.e. *on L and is equal to one on a subset of K of positive capacity.*

Proof. Since L is quasi closed, we can find an open set W with $\gamma(W) < \gamma(K)/2$ such that K and $L - W$ are disjoint closed sets. Now let $v \in C_0^\infty$ be a function satisfying $0 \leqslant v \leqslant 1$ which is equal to 1 near K and 0 near $L - W$; let $\phi \in C_0^\infty$ be equal to 1 near supp v; and let w be an equilibrium potential for W. Then the quasi continuous function $u = v(\phi - w)$ is zero q.e. in L. There is some $\epsilon > 0$ such that the (capacitable) set $K \cap \{u > \epsilon\}$ has positive capacity; for otherwise the (capacitable) set $K \cap \{u > 0\}$ would have zero capacity, which would mean that $w \geqslant 1$ q.e. on K, and hence $\| w \| \geqslant \gamma(K)$. The function $\epsilon^{-1} \min\{u, \epsilon\}$ has the properties required, and the proof is complete.

In the proof of our next theorem we need the following lemma, which is proved in [11, pp. 50–53]. If u is any real-valued function, we let $u^+(x) = \max\{u(x), 0\}$.

LEMMA 2. *If $\{u_n\}$ is a sequence of real-valued functions in C_0^∞ which converges to $u \in W_q^1$ in the norm of W_q^1, then u_n^+ converges to u^+ in the norm of W_q^1.*

THEOREM 4. *Let $Y \subset \mathbf{R}^n$ be open, and let $u \in W_q^1$ be quasi continuous. Then $u \in W_{q0}^1(Y)$ if and only if u vanishes* q.e. *on $\sim Y$.*

Proof. It is clear from Lemma 1 that any quasi continuous function in $W_{q0}^1(Y)$ must vanish q.e. on $\sim Y$, and we must prove the converse. It is known that any function $u \in W_q^1$ is the limit of a sequence of functions $u_n \in W_q^1$ of compact support such that the sets $\{u_n = u\} \nearrow \mathbf{R}^n$; thus we may assume that Y is bounded. We may also assume that the function u is real-valued; moreover, we may assume that $u \geqslant 0$, for in any case we may consider separately the positive part $u^+ = \max\{u, 0\}$ and the negative part $u^- = (-u)^+$. In fact, it is clearly no loss of generality to assume $0 \leqslant u \leqslant 1$. We let B be a fixed open ball containing $\bar{Y}$, and we let $\rho \in C_0^\infty(B)$ be a fixed function such that $\rho = 1$ in a neighborhood of $\bar{Y}$.

Now let $\epsilon > 0$ be arbitrary. It is known that the gradient of a function in W_q^1 must vanish λ — a.e. on the set where the function itself vanishes [13, Theorem 3.2.2(c)]; therefore we can find open sets Ω_0 and Ω such that $\partial Y \subset \Omega_0 \subset \bar{\Omega}_0 \subset \Omega \subset \bar{\Omega} \subset \operatorname{int}\{\rho = 1\}$ and $\int_\Omega |\nabla u|^q \, d\lambda < \epsilon^q$. We can select a function $\psi \in C_0^\infty(\Omega)$ which is equal to some positive constant α on $\bar{\Omega}_0$ and satisfies $\|\psi\| < \epsilon$. We now find a mollifier $\phi \in C_0^\infty(Y \cup \Omega)$ of $u - \psi$ such that $\|\phi - (u - \psi)\| < \epsilon$, and $|\phi - (u - \psi)| < \alpha/2$ except for an open set W with $\gamma_\Omega(W \cap \Omega_0) < \epsilon$. If w is an equilibrium potential for $W \cap \Omega_0$ with respect to Ω, then the function $(\rho - w)\phi \in W_q^1$ is non-positive on an entire neighborhood of ∂Y. We now apply Poincaré's lemma [13], which asserts that there exists a constant $C > 1$ such that

$$\|f\| \leqslant C \|\nabla f\|_{L_q(B)}, \qquad \text{for all} \quad f \in W^1_{q0}(B).$$

We obtain

$$\begin{aligned} \|u - (\rho - w)\phi\| &\leqslant \|u - \phi\| + \|w\phi\| \\ &\leqslant 2\epsilon + C\|\nabla(w\phi)\|_{L_q(\Omega)} \\ &\leqslant 2\epsilon + C\|\nabla w\|_{L_q(\Omega)} + C\|\nabla\phi\|_{L_q(\Omega)} \\ &\leqslant 2\epsilon + C\epsilon + 3C\epsilon \\ &\leqslant 6C\epsilon. \end{aligned}$$

It follows that we can approximate u in the W_q^1 norm by C^∞ functions f such that $f \leqslant 0$ near ∂Y. From Lemma 2 we then can approximate u in the W_q^1 norm by C^∞ functions f such that $f = 0$ near ∂Y. This completes the proof.

THEOREM 5. *Let X be a compact subset of $\mathbf{R}^n$. Then the following assertions are equivalent*:

(b) $\gamma(G - X) = \gamma(G - \operatorname{int} X)$ *for every bounded open set* $G \subset \mathbf{R}^n$.

(b′) *Every quasi continuous function $u \in W_q^1$ which vanishes* q.e. *in $\sim X$ must vanish* q.e. *in ∂X.*

Proof. (b) $\Rightarrow$ (b′). If (b) holds, it follows at once that $\gamma^*(G - X) = \gamma^*(G - \operatorname{int} X)$ for every bounded quasi open set G. Now if u is any quasi continuous function on $\mathbf{R}^n$ which vanishes on $\sim X$, we may apply this fact with $G = \{u \neq 0\}$, and we conclude that u must vanish q.e. on ∂X.

(b′) $\Rightarrow$ (b). Let G be any open set in $\mathbf{R}^n$. If $\gamma(G - X) = \infty$, then (b) is obvious; we therefore assume $\gamma(G - X) < \infty$, and we let w be an equilibrium potential for $G - X$. By redefining w to be equal to 1 of a set of zero exterior capacity (if necessary) we may assume that the sets $\{w < 1\}$ and $\{w = 1\}$ are Borel sets. Now the set $S = \{w = 1\}$ is quasi closed; and by Theorem 2 we have $\gamma(S) \leqslant \| w \|$. Moreover, we have $\gamma(G - \operatorname{int} X - S) = 0$; for otherwise there exists a compact subset of ∂X having positive capacity which is disjoint from the quasi closed set $S \cup [\partial X \cup \sim X - G] \supset \sim X$, so by Theorem 3 we would get a contradiction to (b′). We now have

$$\gamma^*(G - \operatorname{int} X) \leqslant \gamma^*(G - \operatorname{int} X - S) + \gamma^*(S) \leqslant \| w \| = \gamma^*(G - X),$$

which proves (b).

4. Proof of Theorem 1

(b) $\Rightarrow$ (a)

We consider a function $f \in L_q(X)$ which annihilates the rational functions in $L_p(X)$. Then by the Calderon–Zygmund theory [4; 5; 15] we have $u = f * z^{-1} \in W_q^{\,1}$ and $\partial u/\partial \bar{z} = \pi f$ in the sense of distribution theory. Now we may assume that the values of u have been chosen so that u is a quasi continuous function which is identically zero in $\sim X$. The hypothesis guarantees that u vanishes q.e. in ∂X; in particular, u vanishes a.e. in ∂X, and it follows from the elementary theory of Sobolev functions [13] that $f = 0$ a.e. on ∂X. According to Theorem 3 we can find a sequence of functions $\phi_j \in C_0^{\infty}(\operatorname{int} X)$ such that $\phi_j \to u$ in $W_q^{\,1}$. Thus if $h \in L_p(X)$ represents a function which is analytic on int X we have

$$\int_{\operatorname{int} X} h \partial\phi_j/\partial\bar{z} = -\int_{\operatorname{int} X} \phi_j \partial h/\partial \bar{z} = 0$$

and since $\partial\phi_j/\partial\bar{z} \to \partial u/\partial\bar{z} = \pi f$ in $L_q(\operatorname{int} X)$ we conclude

$$\int_X fh = \int_{\partial X} fh + \int_{\operatorname{int} X} fh = 0.$$

This proves (a).

(a) $\Rightarrow$ (b)

It is well-known that every element $T \in W_q^{1*}$ can be realized as a distribution of the form $\tilde{T} = f + \partial g/\partial z + \partial h/\partial \bar{z}$ where $f, g, h \in L_p(\mathbf{R}^2)$; for $u \in C_0^{\infty} \subset W_q^{\,1}$ we have

$$T(u) = \langle \tilde{T}, u \rangle = \int fu \, d\lambda - \int g(\partial u/\partial z) \, d\lambda - \int h(\partial u/\partial \bar{z}) \, d\lambda,$$

where the symbol $\langle , \rangle$ is used for the action of distributions on functions in C_0^∞. From now on we identify T and $\tilde{T}$.

If condition (b) fails, then by Theorem 5 there exists a quasi continuous function $u \in W_q^1$ which is identically zero on $\sim X$ and strictly positive on a set of positive capacity in ∂X; recalling the argument used in the proof of Theorem 3, we can find a quasi continuous function $u \in W_q^1$ which is identically zero on $\sim X$ and identically one on a compact set $K \subset \partial X$ of positive capacity. It is known [12, Lemma 1] that any compact set of positive capacity supports a nonzero element of W_q^{1*}, and hence we can find a distribution $T \in W_q^{1*}$ with support in K and a function $\phi \in C_0^\infty$ such that $T(\phi) = 1$. From the Calderon–Zygmund theory we conclude that $A = T * z^{-1} \in L_p(\mathbf{R}^2, \text{loc})$.

Now let Ω_j be a decreasing sequence of bounded neighborhoods of X with $\bigcap \Omega_j = X$. Then by Theorem 2 we have $\gamma_{\Omega_j}(K) \leqslant \| u \|$ and hence we can find a function $u_j \in C_0^\infty(\Omega_j)$ with $\| u_j \| \leqslant 2 \| u \|$ such that $u_j = 1$ on a neighborhood of K. If now $v \in W_q^1$ is a weak limit of $\{u_j \phi\}$ we have

$$1 = T(\phi) = \lim_j T(u_j \phi) = T(v),$$

but using mollifiers $v_\epsilon \to v$ in W_q^1,

$$\begin{aligned} \pi = T(\pi v) &= \lim_\epsilon T(\pi v_\epsilon) = \lim_\epsilon \langle \partial A / \partial \bar{z}, v_\epsilon \rangle \\ &= -\lim_\epsilon \langle A, \partial v_\epsilon / \partial \bar{z} \rangle = -\lim_\epsilon \int A \, \partial v_\epsilon / \partial \bar{z} \, d\lambda \\ &= -\int A \, \partial v / \partial \bar{z} \, d\lambda. \end{aligned}$$

Therefore $\partial v / \partial \bar{z} \in L_q(X)$ annihilates all rational functions with poles off X, but does not annihilate A.

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